COMBINATORICA 7 (3) (1987) 255—263

DIRECTED CUT TRANSVERSAL PACKING FOR SOURCE-SINK CONNECTED GRAPHS

P. FEOFILOFF and D. H. YOUNGER

Received 4 November 1982

Final version 15 October 1985

Concerning the conjecture that in every directed graph, a maximum packing of directed cut transversals is equal in cardinality to a minimum directed cut, a proof is given for the side coboundaries of a graph. This case includes, and is essentially equivalent to, all source-sink connected graphs, for which Schrijver has given a proof. The method used here first reduces the assertion to a packing theorem for bi-transversals. A packing of bi-transversals of the required size is constructed one edge at a time, by maintaining a Hall-like feasibility condition as each edge is added.

1. Introduction

Transversal Packing Conjecture: For any finite directed graph, a maximum packing of transverals of directed cuts is equal in cardinality to a minimum directed cut.

Woodall [8] describes this as the Menger dual of the directed cut packing minimax equality (Lucchesi and Younger [5], Lovász [4]). This paper gives a proof of the Conjecture for source-sink connected graphs, a proof that builds the required packing of transversals one edge at a time, by maintaining a Hall-like feasibility condition throughout the construction. A proof of this case has been given by Schrijver [7] in a more general context.

Let G be a finite graph with *vertex set* VG and *edge set* eG. The *coboundary operator* δ is a function that takes any subset X of VG to the set δX of edges in G having one end in X and one end in $VG-X$. A *coboundary* in G is any set of edges that lies in the range of δ.

A *directed graph* is a graph in which each edge α is assigned a *positive end* (or tail) $p\alpha$ and a *negative end* (or head) $n\alpha$. A coboundary δX is *directed* if each edge in δX has its positive end in X or if each edge in δX has its negative end in X. A *directed cut* is a minimal nonnull directed coboundary. Let C denote the collection of directed coboundaries of directed graph G.

For a subcollection B of C, a *transversal* of B is a subset of eG that has a nonnull intersection with each nonnull set in B. In the statement of the Conjecture, a transversal of C is called a *transversal of directed cuts*.

AMS subject classification (1980): 05 C 20

For any packing (=disjoint collection) of transversals of C and any nonnull element d in C, $|T| \leqq |d|$. This is elementary. The crux of the Conjecture is that every graph contains a pair T, d of equal size. That is, if eG is a k-transversal of C, then there is in eG a k-packing of transversals of C. A *k-transversal* is a subset r of eG such that $|r \cap d| \geqq k$ for each nonnull d in C. A *k-packing* is a disjoint collection consisting of k elements. Following Seymour [12], we say that transversal r of C *packs* if for the largest integer k such that r is a k-tranversal of C, there is a k-packing T of transversals of C such that $\bigcup T \subseteqq r$. In this terminology, the Conjecture translates to: For each directed graph G, eG is a transversal of C that packs. A natural generalization has been formulated by Edmonds and Giles [2]:

Generalized Conjecture: Every transversal of C packs.

Schrijver [6] has constructed a counterexample to the Generalized Conjecture, but not to the basic Conjecture. The Generalized Conjecture is true for source-sink connected graphs. A directed graph is *source-sink connected* if it is acyclic and each source is joined to each sink by a directed path. A *source* is a vertex of invalence zero; a *sink* is a vertex of outvalence zero.

In this paper, we prove the source-sink connected case of the Generalized Conjecture in terms of side coboundaries of an arbitrary directed graph. We now develop this formulation.

Arguments of a directed coboundary d are defined as follows. The *positive vertex argument* of d is the minimal subset X of VG such that $d = \delta X$, and X contains the positive end of each edge of d. The *positive* (edge) *argument pd* of d is the set of edges of G with positive end in X. The *negative argument nd* is defined dually. Here we refer to *directional duality*, which interchanges the positive and negative ends of each edge.

Let Dn be the collection of elements d in C such that either $d = \emptyset$ or $pd \cap pc \neq \emptyset$ for each nonnull element c of C. Define Dp dually. The union $D = Dp \cup Dn$ is the collection of *side coboundaries* of G. Examples of side coboundaries are given in Figure 1. Let Sn be the collection of n-minimal (=minimal negative argument) elements of $Dn - \{\emptyset\}$; define Sp dually. For an acyclic graph, Sn is the collection of *sink coboundaries*, i.e., coboundaries of the form $\delta\{y\}$, y a sink. A directed coboundary lies in Dn iff its negative vertex argument contains no source vertex. A connected directed graph is source-sink connected iff each directed coboundary is a side coboundary, i.e., $C = D$.

Transversal Packing Theorem: Every transversal of D packs.

This Theorem implies, and is implied by, the source-sink connected case of the Generalized Conjecture.

Our first step in proving this Theorem is a reduction to a Bi-transversal Theorem. For a subset t of eG, let tp denote $t \cap (\bigcup Sp)$. Define tn dually. A *bi-transversal* of D is a set t of edges such that tp is a transversal of Dp and tn is a transversal of Dn. A bi-transversal of D is, in particular, a transversal of D. A *k-bi-transversal* of D is a set r of edges such that rp is a k-transversal of Dp and rn is a k-transversal of Dn. A bi-transversal r of D *packs* if, for the largest integer k such that r is a k-bi-transversal of D, there is in r a k-packing of bi-transversals of D.

Bi-transversal Theorem: Every bi-transversal of D packs.

In Section 3, the Transversal Packing Theorem is reduced to the Bi-transversal Theorem.

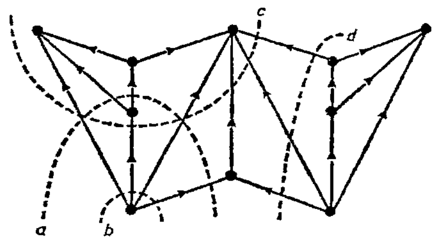

Fig. 1. Side coboundaries a in Dp, b in Sp, c in Dn; directed coboundary d in C—D

2. Meet and join of coboundaries

The domain of the Theorem can be reduced easily to connected graphs. For each directed coboundary d in a connected graph, the following relations hold: $pd \cap nd = d$; $pd \cup nd = eG$ if d is nonnull; $pd \cup nd = \emptyset$ if d is null.

For directed coboundaries c, d in D, if $nc \subseteq nd$, then $nc - c \subseteq nd - d$. Dually, if $pc \subseteq pd$, then $pc - c \subseteq pd - d$. Coboundaries in D are partially ordered by inclusion of positive arguments; a dual partial ordering in D is based on inclusion of negative arguments.

For c in Dn and d in D, there is a coboundary in Dn with negative argument $nc \cap nd$. This is called the *n-meet* of c and d, and is denoted $c \wedge_n d$, or simply $c \wedge d$ if $d \in Dn$. The positive argument is $pc \cup pd$. A consequence of the definition is $c \wedge_n d = c \cap nd \cup d \cap nc$.

For c and d in Dn, there is a coboundary in Dn with negative argument $nc \cup nd$. It is called the *n-join* and is denoted $c \vee_n d$, or simply $c \vee d$. If each of c and d is nonnull, the positive argument is $pc \cap pd$. A consequence is $c \vee_n d = c \cap pd \cup d \cap pc$.

3. Reduction to the Bi-transversal Theorem

We now prove the Transversal Packing Theorem assuming the Bi-transversal Theorem. Let r be a transversal of the collection D of side coboundaries of directed graph G. We interpret the case $D = \{\emptyset\}$ as satisfying the Theorem and assume hereafter that D contains a nonnull element. Let k be the largest integer for which r is a k-transversal of D. Since r is a transversal of D, $k \geqq 1$.

The basis of induction is the case in which r is a k-bi-transversal of D. By the Bi-transversal Theorem, there is in r a k-packing of bi-transversals of D. Each bi-transversal is a transversal and so the assertion holds.

Assume as induction hypothesis that the assertion holds for every graph G' and every subset r' of eG' such that $|r'| < |r|$ or $|r'| = |r|$ and $|eG'| < |eG|$.

Case 1. *For some edge α in r, $r - \{\alpha\}$ is a k-transversal of D.*

By induction hypothesis, there is in $r - \{\alpha\}$ a k-packing of transversals of D. This k-packing satisfies the assertion for r and G.

Case 2. $|r \cap d| = k$ *for some* d *in* $D - Sp \cup Sn$.

Adjust notation so that $d \in Dp$. Let D' be the collection of side coboundaries of the graph G', where G' is obtained from G by contracting the edges of $nd - d$. Let D'' be the collection of side coboundaries of the graph G'' obtained from G by contracting the edges of $pd - d$. Since d is, by hypothesis, not in Sp or Sn, each of $|eG'|$ and $|eG''|$ is strictly smaller than $|eG|$. Now $r' = r \cap pd$ and $r'' = r \cap nd$ are k-transversals of D' and D'', respectively. By induction hypothesis, there is in r' a k-packing T' of transversals of D'. Likewise, there is in r'' a k-packing T'' of transversals of D''. Let T be $\{t' \cup t'' \colon t' \in T',\ t'' \in T'',\ t' \cap d = t'' \cap d\}$. Since $|r \cap d| = k$, each edge of $r \cap d$ lies in exactly one transversal of T' and in one transversal of T''. So T is a k-packing of subsets of r. We assert that each t in T is a transversal of D. This is proved, as in [10], as follows.

Each t in T is of the form $t = t' \cup t''$. Let a be any nonnull element of D, say in Dp. Let $\wedge$ here denote the p-meet, $\vee$ the p-join. If $a \wedge d = \emptyset$, then $t \cap a = t'' \cap a \neq \emptyset$. If $a \wedge d \neq \emptyset$, then $t \cap (a \wedge d) = t' \cap (a \wedge d) \neq \emptyset$ and $t \cap (a \vee d) = t'' \cap (a \vee d) \neq \emptyset$. From the modularity relation $|t \cap (a \wedge d)| + |t \cap (a \vee d)| = |t \cap a| + |t \cap d|$, since $|t \cap d| = 1$, thus $t \cap a \neq \emptyset$. So t is a transversal of Dp. Likewise, t is a transversal of Dn, and thus of all of D. This case is complete.

Case 3. $|r \cap d| > k$ *for each* d *in* $D - Sp \cup Sn$, $r = rp \cup rn$, *and* r *is not a* k*-bi-transversal of* D.

Adjust notation so that rn is not a k-transversal of Dn. There exists a nonnull element a in Dn such that $|rn \cap a| < k$; adjust the choice of a so that it is n-minimal. Since $|r \cap a| \geqq k$, there is an edge β in $(rp - rn) \cap a$. Now a, since it contains an edge of $r - rn$, does not lie in Sn. So there is a nonnull element a' in Dn such that $na' \subseteqq na - \{\beta\}$. Adjust the choice of a' so that it is n-maximal. By the choice of a, $|rn \cap a'| \geqq k$. Let α be any edge in $rn \cap a' - a$. We then have the following properties: for each c in Dn such that $\alpha \in nc$, $\{\alpha, \beta\}$ intersects c; for each d in Dp such that $\beta \in pd$, $\{\alpha, \beta\}$ intersects d.

Let G' be the graph obtained from G by adding a new edge γ with negative end $n\alpha$ and positive end $p\beta$. Let $D' = D'n \cup D'p$ be the collection of side coboundaries of G'. Then $D'n$ is the same as Dn except that γ is added to each c in Dn such that $\alpha \in nc$. And $D'p$ is the same as Dp except that γ is added to each d in Dp such that $\beta \in pd$. Let $r' = (r - \{\alpha, \beta\}) \cup \{\gamma\}$. Since $|r \cap d| \geqq k$ for each d in D, with equality only for d in $Sp \cup Sn$, thus $|r' \cap d| \geqq k$ for each d in D', i.e., r' is a k-transversal of D'. Since $|r'| < |r|$, by induction hypothesis there is in r' a k-packing T' of transversals of D'. Each element of T' that does not contain γ is a transversal of D. For the element t' of T' that contains γ, $t = (t' - \{\gamma\}) \cup \{\alpha, \beta\}$ is a transversal of D. So $T = (T' - \{t'\}) \cup \{t\}$ is a k-packing in r of transversals of D. Again the assertion holds.

This completes the proof of the Transversal Packing Theorem under the assumption of the Bi-transversal Theorem.

4. Constructing bi-transversals one edge at a time

In his proof of Edmonds Disjoint Branchings Theorem [1], Lovász [4] finds one branching that saves enough room in each coboundary for the remaining $k-1$ branchings. This branching is constructed one edge at a time, by successively adding a new edge emanating from the current partial branching. The property required of each new edge is that its choice leaves at least $k-1$ edges unchosen in each coboundary. Our proof also uses this one edge at a time approach; the properties maintained as each new edge is added are called central and feasible.

A subset t of eG is *central in Dn* if each coboundary d in Dn that is disjoint from tn has nd disjoint from tn. Recall that $tn=t\cap(\bigcup Sn)$. *Centrality in Dp* is defined dually. A subset t of eG is *central* if it is central in Dn and in Dp.

The *frontier in Dn* of subset t of eG is the n-maximal coboundary f in Dn that is disjoint from tn. The *frontier in Dp* is defined dually. For t a central subset of k-bi-transversal r, and α any edge of $rp\cup rn$ that lies in the frontier of t in Dn or in the frontier in Dp, subset $t\cup\{\alpha\}$ is also central.

A subcollection C of Dn is *n-disjoint* if $na\cap nb=\emptyset$ for each distinct a, b in C. Define *p-disjoint* dually. For d in Dn, a *constituent* of d is an n-minimal coboundary in $Dn-\{\emptyset\}$ that is a subset of d. The collection of constitutients of d, denoted Cd, is a partition of d that is n-disjoint.

Feasibility is defined next. A *Q-coboundary in Dp* is a p-minimal coboundary q in $Dp-\{\emptyset\}$ such that $rp\cap q\subseteq rn$. Let t be a subset of k-bi-transversal r. Coboundary d in Dn *shades* q in Dp if $rn\cap d\supseteq rp\cap q$. For d in Dn, let Qd be the collection collection of Q-coboundaries in Dp that are disjoint from t and shaded by d. Note that Qd is p-disjoint. Subset t of r is *feasible in Dn* if every coboundary d in Dn satisfies $|(r-t)d|\geqq ld$, where $ld=|Qd|+(k-1)|Cd|$ and $(r-t)d=(rn-tn)\cap d$. *Feasibility in Dp* is defined dually. A subset t of r is *feasible* if it is feasible in Dn and in Dp.

For d in Dn, and subset r of eG, let rd abbreviate $rn\cap d$; for d in Dp, let rd abbreviate $rp\cap d$. Coboundary d in Dn shades q in Dp if $rd\supseteq rq$. Write rQd for $r(\bigcup Qd)$. Note that for d in Dn, $rQd\subseteq rf$, where f is the frontier of t in Dp.

4.1. *Let r be a k-bi-transversal of D. The null set is central and feasible.*

Proof. Centrality of the null set is immediate. For feasibility, consider any coboundary d in D, say in Dn. From $rd\supseteq rQd$ and rp a k-transversal of Dp, $|rd|\geqq|rQd|\geqq k|Qd|$. From rn a k-transversal of Dn, $|rd|\geqq k|Cd|$. Thus $k|rd|=|rd|+(k-1)|rd|\geqq k|Qd|+(k-1)k|Cd|=kld$ and so $|rd|\geqq ld$. ∎

For subset t of r that is central and feasible, an *augment* of t is any edge α of $r-t$ such that $t\cup\{\alpha\}$ is also central and feasible.

Augment Lemma: *Let t be a central feasible subset of k-bi-transversal r of D. If t is not a bi-transversal of D, then t has an augment.*

The Augment Lemma, which is proved in Section 6, implies the Bi-transversal Theorem as follows. Let r be a k-bi-transversal of D. We proceed by induction on k. For $k=1$, the Theorem is trivially true. Assume then that $k\geqq 2$. Let t be a maximal central feasible subset of r. The existence of t follows from 4.1. By the Augment Lemma, t is a bi-transversal of D. Since t is feasible $r-t$ is a $(k-1)$-bi-transversal of D: for any nonnull d in D, $|(r-t)d| \geqq (k-1)|Cd| \geqq k-1$. By induction hypothesis, there is in $r-t$ a $(k-1)$-packing T' of bi-transversals of D. Then $T' \cup \{t\}$ is a k-packing in r of bi-transversals of D. The Theorem follows by induction.

5. Marginal coboundaries and blockers

Let t be a central feasible subset of k-bi-transversal r. A coboundary d in Dp or Dn is *marginal* if $|(r-t)d| = ld$. For edge α in $r-t$, a *blocker in Dn* of α is a marginal coboundary in Dn such that $\alpha \in rd - rQd$. A *blocker in Dp* is defined dually. From the definition of feasible, it follows that $t \cup \{\alpha\}$ is feasible iff α has no blocker either in Dp or in Dn.

5.1. *The l function is supermodular: for d, d' in Dn,*

$$l(d \wedge d') + l(d \vee d') \geqq ld + ld',$$

with equality only if $Q(d \vee d') = Qd \cup Qd'$.

Proof. 1. The cardinality of C is supermodular:

$$|C(d \wedge d')| + |C(d \vee d')| \geqq |Cd| + |Cd'|.$$

To prove this, form a graph G with bipartition (Cd, Cd'), whose edges (a, b) represent the pairs a in Cd, b in Cd', that meet, i.e., $a \wedge b \neq \emptyset$. Now $|C(d \vee d')|$ is equal to the number of components in B; $|C(d \wedge d')|$ is greater than or equal to the number of edges of B; and $|Cd| + |Cd'|$ is equal to the number of vertices of B. The asserted supermodularity relation follows from

$$|eB| + \#\,\text{components}\, B \geqq |VB|,$$

a simple fact about graphs.

2. The cardinality of Q is supermodular. Observe that

$$Q(d \wedge d') = Qd \cap Qd'$$
$$Q(d \vee d') \supseteq Qd \cup Qd'.$$

Adding cardinalities yields

$$|Q(d \wedge d')| + |Q(d \vee d')| \geqq |Qd| + |Qd'|,$$

with equality iff $Q(d \vee d') = Qd \cup Qd'$.

3. Adding to this latter inequality $k-1$ times the inequality for C yields the supermodularity relation for l. ∎

5.2. *If d and d' are marginal coboundaries in Dn, then the meet $d \wedge d'$ and join $d \vee d'$ are also marginal, and $Q(d \vee d') = Qd \cup Qd'$.*

Proof. For d, d' in Dn,

$$d \wedge d' \cap d \vee d' = d \cap d'$$
$$d \wedge d' \cup d \vee d' = d \cup d'.$$

Restrict each equivalence to its edges of $rn - tn$ and add cardinalities:

$$|(r-t)(d \wedge d')| + |(r-t)(d \vee d')| = |(r-t)d| + |(r-t)d'|.$$

From the supermodularity of l and feasibility of t,

$$\begin{aligned}|(r-t)d| + |(r-t)d'| &= ld + ld' \\ &\leqq l(d \wedge d') + l(d \vee d') \\ &\leqq |(r-t)(d \wedge d')| + |(r-t)(d \vee d')|.\end{aligned}$$

Equality holds throughout: $d \wedge d'$ and $d \vee d'$ are marginal and, by 5.1, $Q(d \vee d') = = Qd \cup Qd'$.

5.3. *Each coboundary in $Dn - \{\emptyset\}$ disjoint from tn contains an edge in rn having no blocker in Dp.*

Proof. If an edge has a blocker in Dp, then it must lie in rp. So the assertion holds for coboundary a in Dn unless $ra \subseteqq rp$. Under this condition, adjust the choice of coboundary a so that it is p-minimal, i.e., a Q-coboundary. Let d be a marginal coboundary in Dp that does not shade a; the null coboundary is one such· Adjust the choice of d so that pd is maximal. Let α be any edge of $ra - rd$. Suppose d' is a marginal coboundary in Dp that contains α. By 5.2, $d \vee d'$ is marginal and $Q(d \vee d') = = Qd \cup Qd'$. Now edge α lies in $r(d \vee d')$ but not in pd: the p-maximality of d implies that $d \vee d'$ shades a. So $a \in Q(d \vee d') - Qd \subseteqq Qd'$. Thus d' is not a blocker of α. Indeed, α has no blocker in Dp. ▮

6. Proof of the Augment Lemma

Let t be a central feasible subset of k-bi-transversal r of D. Suppose t is not a bi-transversal of D. Specifically, assume tn is not a transversal of Dn. Then the frontier f of t in Dn is nonnull. For each edge α in rf, $t \cup \{\alpha\}$ is central. We claim that there is an edge in rf that is an augment. It is sufficient that such an edge have no blocker in Dp or in Dn. By 5.3, rf contains an edge α having no blocker in Dp. Suppose that α has a blocker in Dn. Such a blocker is a marginal coboundary in Dn that meets f. Let d be an n-minimal marginal coboundary that meets f. An *internal edge* of coboundary d in Dn is any edge of $rn \cap (nd - d) \cup rQd$. Our second candidate for augment is any internal edge β of d in rf. The existence of β is established in part i of the following proposition.

6.1. *Let q be a coboundary in Dn disjoint from tn. Let d be an n-minimal coboundary in Dn that meets q.*

i) *There is an internal edge of d in rq.*

ii) *Each such edge has no blocker in Dn.*

The proof of 6.1 is given after the main argument is complete.

By part ii of 6.1, β has no blocker in Dn. The only alternative to β an augment is that it have a blocker in Dp, which can happen only if $\beta \in rQd$; let $q_1 = \bigcup Qd$. This blocker is a marginal coboundary d_1 in Dp that contains edge β. Adjust the choice of d_1 so that it is a p-minimal marginal coboundary in Dp that meets q_1. Now q_1 is a coboundary in Dp disjoint from tp; by the dual of 6.1i, with q_1 in the role of q and d_1 in the role of d, there is an internal edge of d_1 in rq_1. Choose any such edge γ; this is our third candidate for augment. It has no blocker in Dp, by 6.1ii (dual). Nor does γ have a blocker in Dn: this follows from our previous appeal to 6.1 by observing that γ is an internal edge of d in rq. So γ is an augment of t.

The proof of the Augment Lemma is complete, except for 6.1.

Proof of 6.1. i) Let c be a coboundary in Cd that meets q. Since $r(c \wedge q) = rc \cap nq \cup \cup rq \cap nc$, either rq intersects $nc - c$, a subset of $nd - d$, and the assertion holds directly, or $r(c \wedge q) \subseteq rc$. Consider the latter alternative. Substitute $c \wedge q$ for c in d, i.e., let $d^- = (d-c) \cup c \wedge q$. Now each coboundary in $Qd - Qd^-$ contains at least one edge of $(r-t)d - (r-t)d^-$:

$$\begin{aligned} |Qd| - |Qd^-| &\leqq |(r-t)\,d| - |(r-t)\,d^-| \\ &\leqq ld - ld^- \\ &\leqq |Qd| - |Qd^-| + (k-1)(|Cd| - |Cd^-|) \\ &\leqq |Qd| - |Qd^-|. \end{aligned}$$

Since d is marginal, so too is d^-, a $|Cd^-| = |Cd|$ whence $|C(c \wedge q)| = |Cc| = 1$. By t central, $c \wedge q$ is disjoint from t, whence $|(r-t)c \wedge q| \geqq k$. By t feasible and d marginal,

$$\begin{aligned} |(r-t)c \wedge g| &= |(r-t)\,\delta| - |(r-t)(d-c)| \\ &\leqq l\delta - l(d-c) \\ &\leqq |Q\delta| - |Q(d-c)| + k - 1, \end{aligned}$$

from which follows $|Q\delta| > |Q(d-c)|$. So $d-c$ does not shade Qd, and thus $r(c \wedge q) \cap \cap rQd \neq \emptyset$. Since $rQd \subseteq rp \cap rn$, there is an internal edge of d in rq.

ii) For any internal edge α of d in rq, consider any marginal coboundary d' in Dn that contains α. By 5.2, $d \wedge d'$ is marginal. Since $\alpha \in nd \cap rd' \subseteq r(d \wedge d')$, thus $d \wedge d'$ meets q. The n-minimality of d implies that $d \wedge d' = d$, whence $nd \subseteq nd'$ and $Q(d \wedge d') = Qd$. So $\alpha \notin nd - d$, whence $\alpha \in rQd \subseteq rQd'$. So d' is not a blocker of α. Indeed, α has no blocker in Dn. ∎

With 6.1, the proof of the Bi-transversal Theorem is complete.

7. Remarks

The above proof does not translate directly to a polynomial algorithm for finding T^*. One that does is given in [11]. It builds a maximum packing T^* for D from maximum packings T^*p and T^*n of transversals of Dp and Dn, each of which is found by an analog of the Lovász algorithm [4] for disjoint branchings.

A reduction to Hall's Theorem approach was used to prove the source-sink connected case of the directed cut packing minimax equality [9]. The Hall's Theorem part of it was there treated by an alternating path approach that yielded a polynomial algorithm. This approach can be used also to relate Gupta's Theorem [3] to Hall's.

Much of this paper was written while the authors were visiting MIT. We thank Kleitman for the opportunity. This research was supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

References

[1] J. EDMONDS, Edge-disjoint branchings, *in: Combinatorial Algorithms,* (R. Rustin, ed.) Algorithmics Press, New York, 1973, 91—96.

[2] J. EDMONDS and R. GILES, A min-max relation for submodular functions on graphs, *in: Studies in integer programming* (P. L. Hammer, et al, eds.) Annals of Discrete Math. **1** (1977), 185—204.

[3] R. P. GUPTA, A decomposition theorem for bipartite graphs, *in: Theory of Graphs* (P. Rosenstiehl, ed.) Gordon and Breach, New York, 1967, 135—138.

[4] L. LOVÁSZ, On two minimax theorems in graph theory, *J. Combinatorial Theory (B)* **21** (1976), 96—103.

[5] C. L. LUCCHESI and D. H. YOUNGER, A minimax relation for directed graphs, *J. London Math. Soc. (2)* **17** (1978), 369—374.

[6] A. SCHRIJVER, A counterexample to a conjecture of Edmonds and Giles, *Discrete Math.* **32** (1980), 213—214.

[7] A. SCHRIJVER, Min-max relations for directed graphs, *Annals of Discrete Math.* **16** (1982), 261—280.

[8] D. R. WOODALL, Menger and König systems, *in: Theory and Applications of Graphs* (Proceedings of the Western Michigan Graph Theory Conference 1976) Springer-Verlag Lecture Notes in Math. **642** (1978), 620—635.

[9] D. H. YOUNGER, From shortest paths to directed cut transversals, *Annals of the New York Academy of Sciences,* **319** (1979), 555—564.

[10] D. H. YOUNGER, Maximum families of disjoint directed cut sets, *in: Recent Progress in Combinatorics* (W. T. Tutte, ed.) Academic Press, 1969, 329—333.

[11] P. FEOFILOFF, *Disjoint Transversals of Directed Coboundaries,* Ph. D. Thesis, University of Waterloo, 1983.

[12] P. D. SEYMOUR, The matroids with the max-flow min-cut property, *J. Combinatorial Theory (B)* **23** (1977), 189—222.

P. Feofiloff, D. H. Younger

Department of Combinatorics and Optimatization
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1